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# APPROXIMATE METHOD FOR DETERMINING THE MAXIMUM TEMPERATURE DURING QUASISTATIONARY HEATING OF A PIECEWISE-HOMOGENEOUS HALF-SPACE

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*A method is proposed to calculate the maximum temperature of the surface of a piecewise-homogeneous half-space heated by a uniformly moving, locally distributed heat flow. Analytical solutions of the corresponding quasistationary heat-conduction problems are obtained for small and large values of the Peclet number. These solutions are used to derive formulas for calculating the maximum temperature in the case of intermediate (moderate) values of the Peclet number.*

**Key words:** *temperature, heat conduction, fast heating.*

**Introduction.** Increased interest in solving quasistationary heat-conduction problems is motivated by the formulation of thermal friction problems in [1–5] and studies of other authors. In such formulations, a moving heat flow distributed in the contact area is specified on the working surface of each of the elements of the friction pair. The intensity of this frictional heat flow it is equal to the specific friction power — the product of the relative sliding velocity of the bodies by the shear stress. The latter, in turn, is proportional to the pressure obtained as a result of solution of the corresponding contact problem. In this case, it is customary to use a uniform or elliptic (Hertz) distribution of the contact pressure.

This computational scheme is used, in particular, to determine the flash temperature at the sites of frictional contact between the surface protrusions of rubbing bodies [6]. Because the spots of contact have small sizes, the corresponding thermal friction problems are formulated for a semi-infinite body (half-space) whose surface is subjected to a uniformly moving frictional heat flow specified in a bounded region [2, 4, 7, 8]. Analytical solutions of such problems have been obtained for two limiting values of the sliding velocity: stationary and high-velocity. In the latter case, in the heat-conduction equation, the second derivative of the temperature with respect to the independent variable in the sliding direction is ignored [5]. In the case of intermediate (moderate) values of the Peclet number (Pe), solutions are obtained using interpolation methods based on constructing *a priori* formulas, which in particular cases coincide with the well-known stationary and high-velocity solutions [9, 10].

The approach described above is used in the present work to obtain a solution of the three-dimensional quasistationary thermal-conduction problem for a piecewise-homogeneous body consisting of a layer applied onto the surface of a half-space. In solving thermal friction problems, the main objective is to obtain a restriction on the maximum permissible temperature level; therefore, emphasis is placed on constructing engineering formulas for calculating the maximum temperature of compound bodies.

**1. Formulation of the Problem.** We consider a piecewise-homogeneous half-space which consists of a layer of finite thickness $d$ applied onto the surface of a semi-infinite body. A heat flow distributed with intensity $q$ in a square with side $a$ moves at constant velocity $V$ on the free surface of the layer (Fig. 1). We assume that the thermal contact of the bodies is perfect and that outside the heating region, the surface of the layer is thermally insulated.

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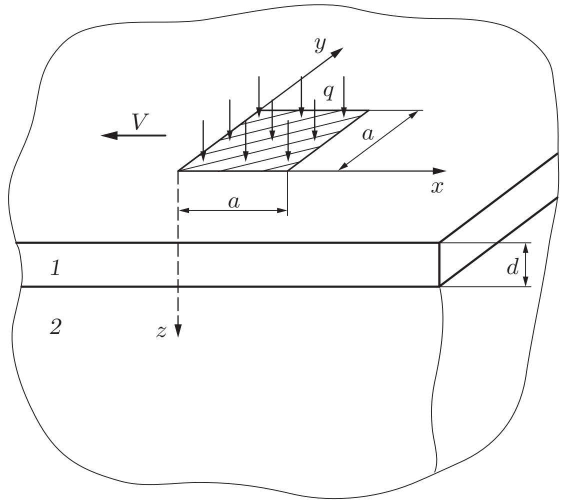


Fig. 1. Diagram of heating of the compound body: 1) layer; 2) surface of the semi-infinite body.

Let us introduce a rectangular coordinate system $Oxyz$ with the positive direction of the $Ox$ axis chosen along the lower side of the square opposite to the direction of motion of the heat flow, the positive direction of the axis $Oy$ along the left side of the square, and the positive $Oz$ direction normal to the surface inward the piecewise-homogeneous body. The subscripts 1 and 2 denote the quantities that refer to the layer and half-space, respectively.

Let us construct a solution of the quasihomogeneous heat-conduction problem:

$$\frac{\partial^2 T_j}{\partial x^2} + \frac{\partial^2 T_j}{\partial y^2} + \frac{\partial^2 T_j}{\partial z^2} = \frac{V}{k_j}\,\frac{\partial T_j}{\partial x}, \qquad j = 1, 2; \tag{1.1}$$

$$-K_1\,\frac{\partial T_1}{\partial z}\Big|_{z=0} = \begin{cases} q(x,y), & 0 \leqslant x, y \leqslant a, \\ 0, & (-\infty < x, y < 0) \cup (a < x, y < \infty); \end{cases} \tag{1.2}$$

$$K_1\,\frac{\partial T_1}{\partial z}\Big|_{z=d} = K_2\,\frac{\partial T_2}{\partial z}\Big|_{z=d}, \qquad -\infty < x, y < \infty; \tag{1.3}$$

$$T_1(x,y,d) = T_2(x,y,d); \tag{1.4}$$

$$\{T_1, T_2\} \to 0 \qquad \text{at} \quad \sqrt{x^2+y^2+z^2} \to \infty. \tag{1.5}$$

Here $T_j$ is the temperature, $q$ is the heat-flow intensity, and $K_j$ and $k_j$ are the thermal conductivity and temperature diffusivity, respectively.

In equalities (1.1)–(1.5), converting to the dimensionless variables and parameters

$$X = \frac{x}{a}, \qquad Y = \frac{y}{a}, \qquad Z = \frac{z}{a}, \qquad \delta = \frac{d}{a}, \qquad K^* = \frac{K_2}{K_1}; \tag{1.6}$$

$$Q(X,Y) = \frac{q(x,y)}{q_0}, \qquad \mathrm{Pe}_j = \frac{Va}{k_j}, \qquad \theta_j = \frac{T_j K_j}{q_0 a}, \qquad j = 1, 2$$

($q_0$ is the maximum intensity of the heat flow), we obtain

$$\frac{\partial^2 \theta_j}{\partial X^2} + \frac{\partial^2 \theta_j}{\partial Y^2} + \frac{\partial^2 \theta_j}{\partial Z^2} = \mathrm{Pe}_j\,\frac{\partial \theta_j}{\partial X}, \qquad j = 1, 2,$$

$$\frac{\partial \theta_1}{\partial Z}\Big|_{Z=0} = \begin{cases} -Q(X,Y), & 0 \leqslant X, Y \leqslant 1, \\ 0, & (-\infty < X, Y < 0) \cup (1 < X, Y < \infty), \end{cases}$$

$$\left.\frac{\partial\theta_1}{\partial Z}\right|_{Z=\delta} = K^*\left.\frac{\partial\theta_2}{\partial Z}\right|_{Z=\delta}, \qquad -\infty < X, Y < \infty, \tag{1.7}$$

$$\theta_1(X,Y,\delta) = \theta_2(X,Y,\delta), \qquad -\infty < X, Y < \infty,$$

$$\{\theta_1,\theta_2\} \to 0 \qquad \text{at} \quad \sqrt{X^2+Y^2+Z^2} \to \infty.$$

**2. Solution of the Problem.** In the boundary-value problem (1.7), we apply the double Fourier integral transformation to the dimensionless variables $X$ and $Y$ [11]:

$$\bar{\bar{\theta}}_j(\xi,\eta,Z) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\theta_j(X,Y,Z)\exp\left[i(\xi X+\eta Y)\right]dX\,dY.$$

As a result, we obtain

$$\frac{d^2\bar{\bar{\theta}}_j}{dZ^2} - æ_j^2\bar{\bar{\theta}}_j = 0, \qquad j=1,2; \tag{2.1}$$

$$\left.\frac{d\bar{\bar{\theta}}_1}{dZ}\right|_{Z=0} = -\bar{\bar{Q}}(\xi,\eta); \tag{2.2}$$

$$\left.\frac{d\bar{\bar{\theta}}_1}{dZ}\right|_{Z=\delta} = K^*\left.\frac{d\bar{\bar{\theta}}_2}{dZ}\right|_{Z=\delta}; \tag{2.3}$$

$$\bar{\bar{\theta}}_1(\xi,\eta,\delta) = \bar{\bar{\theta}}_2(\xi,\eta,\delta); \tag{2.4}$$

$$\bar{\bar{\theta}}_2 \to 0 \qquad \text{at} \quad Z\to\infty, \tag{2.5}$$

where

$$\bar{\bar{Q}}(\xi,\eta) = \frac{1}{2\pi}\int\limits_0^1\int\limits_0^1 Q(X,Y)\exp\left[i(\xi X+\eta Y)\right]dX\,dY; \tag{2.6}$$

$$æ_j^2 = \omega^2 - i\xi\,\mathrm{Pe}_j, \quad j=1,2, \qquad \omega^2 = \xi^2+\eta^2. \tag{2.7}$$

The general solution of the differential equations (2.1) has the form

$$\bar{\bar{\theta}}_j(\xi,\eta,Z) = A_j\exp(-æ_j Z) + B_j\exp(æ_j Z), \tag{2.8}$$

where $A_j(\xi,\eta)$ and $B_j(\xi,\eta)$ ($j=1,2$) are unknown functions of the transformation parameters. The regularity condition (2.5) implies that $B_2=0$. Substituting solution (2.8) into the remaining boundary conditions (2.2)–(2.4), we arrive at the following system of three linear algebraic equations for the unknown functions $A_1$, $A_2$, and $B_1$:

$$æ_1(A_1-B_1) = \bar{\bar{Q}}(\xi,\eta),$$

$$\bar{\bar{\lambda}}A_1\exp(-æ_1\delta) - \bar{\bar{\lambda}}B_1\exp(æ_1\delta) - A_2K^*\exp(-æ_2\delta) = 0;$$

$$A_1\exp(-æ_1\delta) + B_1\exp(æ_1\delta) - A_2\exp(-æ_2\delta) = 0. \tag{2.9}$$

Here $\bar{\bar{\lambda}} = \sqrt{æ_1/æ_2}$.

The maximum temperature of the piecewise-homogeneous body is reached on the surface of the layer. Therefore, we write the solution of the system of linear equations (2.9) only for the functions $A_1$ and $B_1$ included in the transform $\bar{\bar{\theta}}_1$:

$$A_1 = -\bar{\bar{Q}}(K^*+\bar{\bar{\lambda}})\exp(æ_1\delta)/D, \qquad B_1 = \bar{\bar{Q}}(K^*-\bar{\bar{\lambda}})\exp(-æ_1\delta)/D;$$

$$D = æ_1[(\bar{\bar{\lambda}}-K^*)\exp(-æ_1\delta) - (\bar{\bar{\lambda}}+K^*)\exp(æ_1\delta)]. \tag{2.10}$$

Substituting relations (2.10) for $Z = 0$ into solution (2.8), we obtain

$$\bar{\bar{\theta}}_1(\xi,\eta,0) = \bar{\bar{\theta}}_0(\xi,\eta,0)\,\frac{1-\gamma(\xi,\eta)}{1+\gamma(\xi,\eta)} = \bar{\bar{\theta}}_0(\xi,\eta,0)\Big[1 + 2\sum_{n=1}^{\infty}(-1)^n\gamma^n(\xi,\eta)\Big]; \tag{2.11}$$

$$\bar{\bar{\theta}}_0(\xi,\eta,0) = \bar{\bar{Q}}(\xi,\eta)/æ_1, \qquad \gamma(\xi,\eta) = \bar{\bar{C}}\exp(-2æ_1\delta), \qquad \bar{\bar{C}} = (K^* - \bar{\bar{\lambda}})/(K^* + \bar{\bar{\lambda}}). \tag{2.12}$$

The form of relation (2.11) indicates that in the space of the originals, the dimensionless maximum temperature of the layer surface can be written as

$$\theta_{1,\max} = \theta_{0,\max}[1 + F(C,\delta,\mathrm{Pe}_1)], \tag{2.13}$$

where $\theta_{0,\max}$ is the dimensionless temperature of the surface of the half-space with the thermophysical properties of the layer; $-1 < C < 1$ is a dimensionless parameter whose Fourier transform is defined by formula (2.12). The function $F(C,\delta,\mathrm{Pe}_1)$ in relation (2.13) is unknown. However, taking into account formulas (2.11) and (2.12), it is possible to construct its asymptotic expressions for the limiting values of the dimensionless layer thickness $\delta$:

$$F(C,\delta,\mathrm{Pe}_1) = \begin{cases} 2\displaystyle\sum_{n=1}^{\infty}(-C)^n = -\frac{2C}{1+C}, & \delta \to 0, \\ 0, \delta \to \infty \end{cases} \tag{2.14}$$

or

$$F(C,\delta,\mathrm{Pe}_1) = -\frac{2C}{1+C}\,F^*(C,\delta,\mathrm{Pe}_1), \qquad F^*(C,\delta,\mathrm{Pe}_1) = \begin{cases} 1, \delta \to 0, \\ 0, \delta \to \infty. \end{cases} \tag{2.15}$$

In view of (2.14) and (2.15), from equality (2.13) we obtain

$$F^*(C,\delta,\mathrm{Pe}_1) = \frac{1+C}{2C}\Big(1 - \frac{\theta_{1,\max}}{\theta_{0,\max}}\Big). \tag{2.16}$$

Thus, to determine the maximum temperature of the homogeneous body, it is necessary to know the function $F^*(C,\delta,\mathrm{Pe}_1)$. Because of the complex structure of the function $\gamma(r,s)$ in (2.12), this problem cannot be solved for arbitrary velocity of the heat flow. Therefore, we first construct solutions for the two limiting modes: stationary heating ($V \approx 0$ and $0 \leqslant \mathrm{Pe}_1 < 0.4$) and fast heating ($\mathrm{Pe}_1 > 20$) [12].

**3. Stationary Heat-Conduction Problem.** In this case, $\mathrm{Pe}_1 = \mathrm{Pe}_2 = 0$ and formulas (2.7) imply that $\sqrt{\omega^2 - i\,\mathrm{Pe}_1} = |\omega| = \sqrt{\xi^2 + \eta^2}$ as $\lambda \to 1$. The parameter $C$ in (2.12) becomes

$$C = (K^* - 1)/(K^* + 1). \tag{3.1}$$

From relations (2.11), we obtain

$$\bar{\bar{\theta}}_1(\xi,\eta,0) = \bar{\bar{\theta}}_0(\xi,\eta,0) + 2\sum_{n=1}^{\infty}(-C)^n\,\bar{\bar{G}}_n(\xi,\eta,0); \tag{3.2}$$

$$\bar{\bar{\theta}}_0(\xi,\eta,0) = \bar{\bar{Q}}(\xi,\eta)/|\omega|, \qquad \bar{\bar{G}}_n(\xi,\eta,0) = (\bar{\bar{Q}}(\xi,\eta)/|\omega|)\exp(-2n\delta|\omega|). \tag{3.3}$$

We consider in more detail the heating of the layer surface by a uniform heat flow of intensity $q(x,y) = q_0$ ($Q(X,Y) = 1$). Taking into account the value of the integral (see [13])

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\exp(-2n\delta|\omega|)}{|\omega|}\,\exp[-i(\xi X + \eta Y)]\,d\xi\,d\eta = \frac{1}{\sqrt{X^2 + Y^2 + 4n^2\delta^2}}$$

and using the convolution theorem for the Fourier integral transformation, from relations (3.3) we obtain

$$G_n(X,Y,0) = \frac{1}{2\pi}\int_0^1\int_0^1 \frac{dX'\,dY'}{\sqrt{(X-X')^2 + (Y-Y')^2 + 4n^2\delta^2}}. \tag{3.4}$$

Because of the symmetry of the problem, the maximum temperature is reached at the center of the heating region on the layer surface. After integration for $X = Y = 0.5$, formula (3.4) becomes

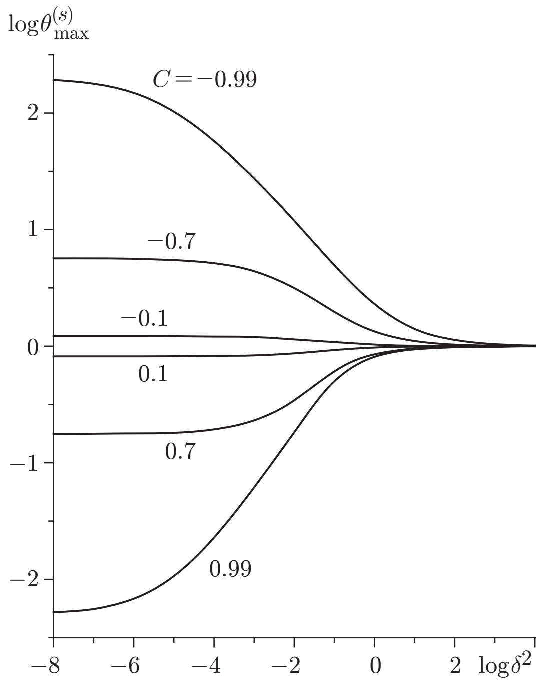


Fig. 2

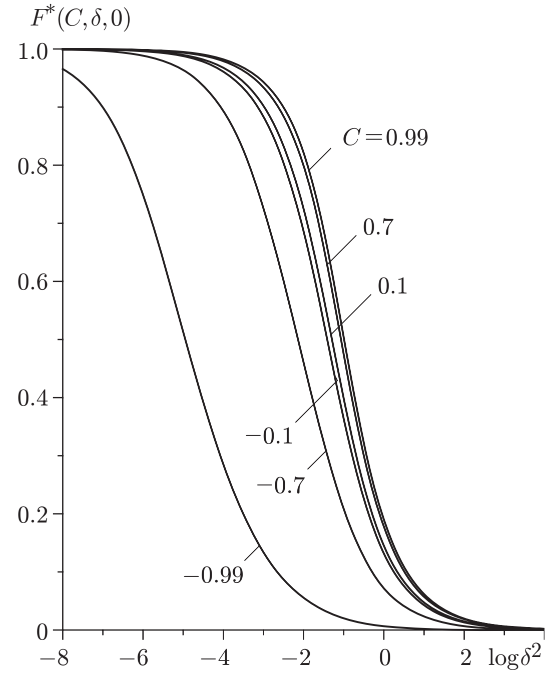


Fig. 3

Fig. 2. Ratio of the maximum stationary temperatures of the piecewise-homogeneous and inhomogeneous half-spaces versus dimensionless layer thickness.

Fig. 3. Function $F^*(C, \delta, 0)$ versus relative layer thickness for various values of the parameter $C$.

$$G_{n,\max} \equiv G_n\Big(\frac{1}{2}, \frac{1}{2}, 0\Big) = \frac{1}{\pi}\Big[\ln\Big|\frac{\sqrt{2+16n^2\delta^2}+1}{\sqrt{2+16n^2\delta^2}-1}\Big| - 4n\delta\arctan\,\frac{1}{4n\delta\sqrt{2+16n^2\delta^2}}\Big]. \tag{3.5}$$

Setting $\delta = 0$ in equalities (3.3), we obtain $\bar{\bar{\theta}}_0(\xi, \eta, 0) = \bar{\bar{G}}_n(\xi, \eta, 0)$. Then, from equality (3.5), we obtain the known value of the dimensionless maximum temperature of the homogeneous half-space [4]

$$\theta_{0,\max} = \frac{1}{\pi}\,\ln\Big|\frac{\sqrt{2}+1}{\sqrt{2}-1}\Big| = \frac{2}{\pi}\,\ln|\sqrt{2}+1| \approx \frac{1}{\sqrt{\pi}}. \tag{3.6}$$

The maximum dimensionless temperature of the piecewise-homogeneous body is found from the solution of (3.2):

$$\theta^{(s)}_{1,\max} = \theta^{(s)}_{0,\max} + 2\sum_{n=1}^{\infty}(-C)^n G_{n,\max}. \tag{3.7}$$

Here $\theta^{(s)}_{0,\max}$ is calculated by formula (3.6) and $G_{n,\max}$ by formula (3.5); the superscript $(s)$ indicates that the solution considered is stationary.

Substituting Eq. (3.7) into (2.16), we obtain

$$F^*(C, \delta, 0) = -\frac{1+C}{C\theta^{(s)}_{0,\max}}\sum_{n=1}^{\infty}(-C)^n G_{n,\max}. \tag{3.8}$$

Curves of the ratio $\theta^{(s)}_{\max} = \theta^{(s)}_{1,\max}/\theta^{(s)}_{0,\max}$ versus dimensionless layer thickness $\delta$ for various values of the dimensionless parameter $C$ in (3.1) are shown in Fig. 2. We note that for $C \to -1$ ($K^* \to 0$), the thermal conductivity of the layer is much higher that the thermal conductivity of the foundation ($K_1 \gg K_2$), and, in contrast, for $C \to 1$ ($K^* \to \infty$), we have $K_1 \ll K_2$. As the parameter $C$ increases with fixed thickness of the layer $\delta$, the maximum temperature decreases. The highest temperature is reached in the case of a heat-conducting layer applied onto a thermally insulated half-space. If the thermal conductivity of the layer is higher (lower) than that of the foundation, an increase in the layer thickness leads to a decrease (increase) in the temperature. For $\delta > 3.5$ in both cases, one can ignore the effect of the layer on the maximum temperature of the piecewise-homogeneous body and use solution (3.6) for a homogeneous half-space in the calculations.

In view of equality (2.16), the maximum temperature can also be determined using the function $F^*(C, \delta, 0)$ by the formula

$$\theta_{1,\max}^{(s)} = \theta_{0,\max}^{(s)}\Big[1 - \frac{2C}{1+C} F^*(C, \delta, 0)\Big], \qquad \theta_{0,\max}^{(s)} = \frac{1}{\sqrt{\pi}}. \tag{3.9}$$

In (3.8), the function $F^*(C, \delta, 0)$ decreases monotonically with increase in the parameter $\delta$ (Fig. 3). A decrease in the relative thermal conductivity of the layer $K^*$ [the parameter $C$ in (3.1)] with its fixed thickness $\delta$ leads to a decrease in the values of the function $F^*(C, \delta, 0)$. Knowing the thermophysical ($C$) and geometrical ($\delta$) parameters of the problem, we find the value of the function $F^*(C, \delta, 0)$ from the corresponding curve in Fig. 3. Substituting it into formula (3.9), we obtain the maximum temperature of the piecewise-homogeneous body.

Let us also consider the case of heating of the surface of a piecewise-homogeneous body by a heat flow distributed uniformly over a circle of radius $0.5a$ on the boundary surface. The solution of the corresponding stationary axisymmetric heat-conduction problem is obtained using the zero-order Hankel integral transformation with respect to the dimensionless variable $\rho = 2r/a$, where $r$ is the radial component of the cylindrical coordinate system $(r, \varphi, z)$ with origin at the center of the heating circle.

The maximum temperature in this case is determined from formula (3.7) for

$$G_{n,\max} = \int\limits_0^\infty \frac{\exp(-n\delta\xi)}{\xi} J_1(\xi)\, d\xi = \sqrt{n^2\delta^2 + 1} - n\delta, \qquad \theta_{0,\max} = \int\limits_0^\infty \frac{J_1(\xi)}{\xi}\, d\xi = 1$$

[$J_1(\xi)$ is the first-order Bessel function of the first kind].

**4. Fast Heating.** The fast heating regime occurs for $\mathrm{Pe}_1 \geqslant 20$ and is characterized by the fact that the temperature-gradient variation in the $x$ and $y$ directions ($\partial^2 T_j/\partial x^2$ and $\partial^2 T_j/\partial y^2$) in the heat-conduction equation (1.1) can be ignored [14]. Then, from relation (2.1) it follows that $\omega^2 \to 0$ and $\sqrt{\omega^2 - i\,\mathrm{Pe}_1} \to \sqrt{-i\,\mathrm{Pe}_1}$, and the parameter $C$ in (2.12) becomes

$$C = \frac{K^* - \lambda}{K^* + \lambda}, \qquad \lambda = \sqrt{\frac{\mathrm{Pe}_1}{\mathrm{Pe}_2}} = \sqrt{\frac{k_2}{k_1}}. \tag{4.1}$$

Thus, we have a two-dimensional (in the variables $X$ and $Z$) quasistationary heat-conduction problem for a piecewise-homogeneous half-space heated in a zone $0 \leqslant X \leqslant 1$ on the surface $Z = 0$ by a distributed heat flow $Q(X) = q(aX)/q_0$. Solution of this problem using the Fourier integral transformation with respect to the variable $X$ yields the temperature transform on the layer surface $Z = 0$:

$$\overline{\theta}_1(\xi, 0) = \overline{\theta}_0(\xi, 0)\Big[1 + 2\sum_{n=1}^{\infty}(-1)^n \gamma^n(\xi)\Big],$$

$$\overline{\theta}_0(\xi, 0) = \overline{Q}(\xi)/\sqrt{-i\xi\,\mathrm{Pe}_1}, \qquad \gamma(\xi) = C\exp\big(-2\delta\sqrt{-i\xi\,\mathrm{Pe}_1}\,\big), \tag{4.2}$$

$$\overline{Q}(\xi) = \frac{1}{\sqrt{2\pi}} \int\limits_0^1 Q(X)\exp(i\xi X)\, dX.$$

Using the convolution theorem for the integral Fourier transformation, from relations (4.2) we obtain

$$\theta_1(X, 0) = \frac{1}{\sqrt{2\pi}} \int\limits_0^1 Q(X') \int\limits_{-\infty}^{\infty} \frac{1}{\sqrt{-i\xi\,\mathrm{Pe}_1}} \Big[1 + 2\sum_{n=1}^{\infty}(-C)^n \exp\big(-2n\delta\sqrt{-i\xi\,\mathrm{Pe}_1}\,\big)\Big] \exp\big(-i\xi(X - X')\big)\, d\xi\, dX'. \tag{4.3}$$

Taking into account the value of the integral (see [13])

$$\int\limits_{-\infty}^{\infty} \frac{\exp(-2n\delta\sqrt{-i\xi\,\mathrm{Pe}_1})}{\sqrt{-i\xi\,\mathrm{Pe}_1}} \exp(-i\xi X)\, d\xi = \sqrt{\frac{\pi}{\mathrm{Pe}_1 X}} \exp\Big(-\frac{n^2\delta^2\,\mathrm{Pe}_1}{X}\Big), \qquad n \geqslant 0,$$

from equality (4.3) we obtain the following computational formula for the dimensionless temperature on the surface of the piecewise-homogeneous half-space:

$$\theta_1(X,0) = \begin{cases} 0, & X \leqslant 0, \\ \dfrac{1}{2\sqrt{\pi\,\mathrm{Pe}_1}} \displaystyle\int_0^X Q(X')H(X-X')\,dX', & 0 \leqslant X \leqslant 1, \\ \dfrac{1}{2\sqrt{\pi\,\mathrm{Pe}_1}} \displaystyle\int_0^1 Q(X')H(X-X')\,dX', & X \geqslant 1, \end{cases} \tag{4.4}$$

where

$$H(X) = \frac{1}{\sqrt{X}}\Big[1 + 2\sum_{n=1}^{\infty}(-C)^n \exp\Big(-\frac{n^2\delta^2 P_1}{X}\Big)\Big]. \tag{4.5}$$

In the case of constant intensity of the heat flow $Q(x) = 1$, using the value of the integral

$$\int_0^X \frac{1}{\sqrt{X-X'}} \exp\Big(\frac{-n^2\delta^2\,\mathrm{Pe}_1}{X-X'}\Big)\,dX'$$

$$= 2\sqrt{X}\exp\Big(-\frac{n^2\delta^2\,\mathrm{Pe}_1}{X}\Big) - 2n\delta\sqrt{\pi\,\mathrm{Pe}_1}\,\mathrm{erfc}\Big(\frac{n\delta\sqrt{\mathrm{Pe}_1}}{\sqrt{X}}\Big), \qquad n \geqslant 0$$

from relations (4.4) and (4.5) we obtain

$$\theta_1(X,0) = \theta_0(X,0)\Big[1 + 2\sum_{n=1}^{\infty}(-C)^n R_n(X)\Big], \qquad 0 \leqslant X \leqslant 1,$$

$$\theta_0(X,0) = \sqrt{\frac{X}{\pi\,\mathrm{Pe}_1}}, \qquad R_n(X) = \exp\Big(-\frac{n^2\delta^2\,\mathrm{Pe}_1}{X}\Big) - n\delta\sqrt{\frac{\pi\,\mathrm{Pe}_1}{X}}\,\mathrm{erfc}\Big(n\delta\sqrt{\frac{\mathrm{Pe}_1}{X}}\Big). \tag{4.6}$$

The maximum value of the temperature is reached at the right (exit) end of the heating zone [12]. Then, formulas (4.6) for $X = 1$ imply that

$$\theta^{(f)}_{1,\max} \equiv \theta_1(1,0) = \theta^{(f)}_{0,\max}\Big[1 + 2\sum_{n=1}^{\infty}(-C)^n R_{n,\max}\Big] \tag{4.7}$$

$$\theta^{(f)}_{0,\max} = 1/\sqrt{\pi\,\mathrm{Pe}_1}, \qquad R_{n,\max} = \exp(-n^2\delta^2\,\mathrm{Pe}_1) - n\delta\sqrt{\pi\,\mathrm{Pe}_1}\,\mathrm{erfc}\,(n\delta\sqrt{\mathrm{Pe}_1}\,), \tag{4.8}$$

where the superscript $(f)$ indicates that the fast heating regime is considered. We note that for the maximum dimensionless temperature $\theta^{(f)}_{0,\max}$ of a homogeneous half-space, formula (4.8) was first obtained in [14].

Solution (4.7), (4.8) can also be written as

$$\theta^{(f)}_{1,\max} = \theta^{(f)}_{0,\max}\Big[1 - \frac{2C}{1+C}\,F^*(C,\delta,\mathrm{Pe}_1)\Big], \quad F^*(C,\delta,\mathrm{Pe}_1) = -\frac{1+C}{C\theta^{(f)}_{0,\max}}\sum_{n=1}^{\infty}(-C)^n R_{n,\max}. \tag{4.9}$$

For $\delta \to 0$, formula (4.8) implies that $R_{n,\max} \to 1$, and from relation (4.9), we obtain $F^*(C,0,\mathrm{Pe}_1) = 1$. If $\delta \to \infty$, then $R_{n,\max} \to 0$ and $F^*(C,\infty,\mathrm{Pe}_1) \to 0$. Thus, the function $F^*(C,\delta,\mathrm{Pe}_1)$ decreases monotonically with increase in the parameter $\delta$ and takes values between zero and unity.

The dependence of the ratio $\theta^{(f)}_{\max} = \theta^{(f)}_{1,\max}/\theta^{(f)}_{0,\max}$ on the dimensionless complex $\delta^2\,\mathrm{Pe}_1$ is shown in Fig. 4. The same dependence for the function $F(C,\delta,\mathrm{Pe}_1)$ in (4.9) is given in Fig. 5 [the parameter $C$ is calculated from (4.1)]. It is obvious that for $\delta^2\,\mathrm{Pe}_1 > 1$ ($\delta > 1/\sqrt{\mathrm{Pe}_1}$), the effect of the layer on the maximum temperature of the piecewise-homogeneous body can be ignored. At the same time, this indicates that the effective heating depth (the distance from the layer surface at which the temperature is 5% of the maximum surface temperature) decreases with increase in the velocity of motion of the heat flow.

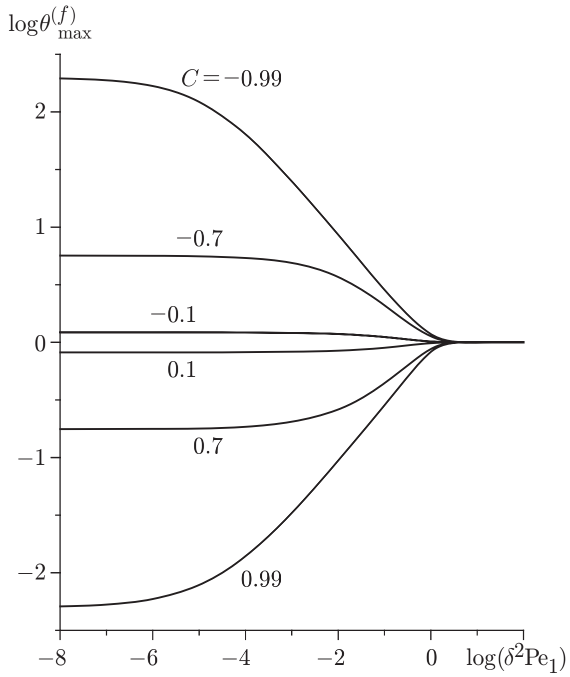


Fig. 4

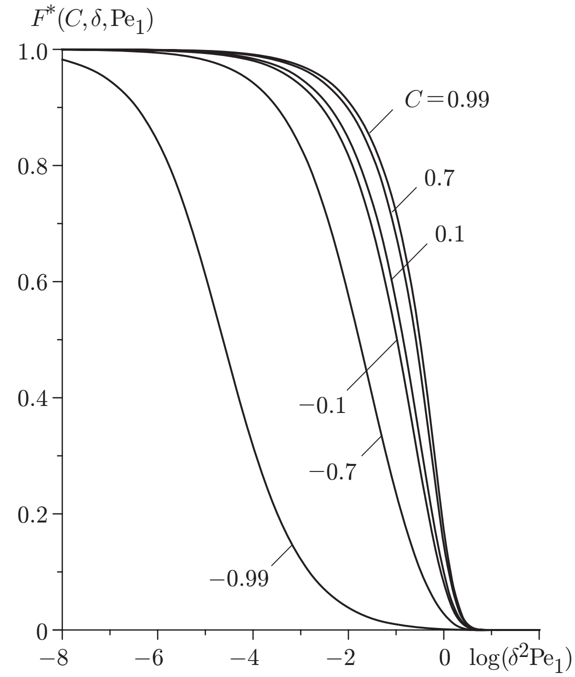


Fig. 5

Fig. 4. Ratio of the maximum temperatures of piecewise-homogeneous and inhomogeneous half-spaces versus dimensionless layer thickness for fast heating.

Fig. 5. Function $F^*(C, \delta, \mathrm{Pe}_1)$ versus increasing dimensionless complex $\delta^2\,\mathrm{Pe}_1$ for various values of the parameter $C$.

**5. Solution for the Intermediate Values of the Peclet Number ($0.4 \leqslant \mathrm{Pe}_1 \leqslant 20$).** In the cases of stationary and fast heating, heat transfer in the solid occurs primarily by the conductive and convective mechanisms, respectively [8]. Under the assumption that for the intermediate values of the heat-flow velocity, these types of heat transfer operate in parallel and by analogy with the formula for the electric-current resistance for parallel conductors, Archard [9] proposed the following *a priori* interpolation formula for the maximum temperature of a homogeneous half-space for intermediate values of the Peclet number $0.4 \leqslant \mathrm{Pe}_1 \leqslant 20$:

$$1/\theta_{0,\max} = 1/\theta^{(s)}_{0,\max} + 1/\theta^{(f)}_{0,\max}. \tag{5.1}$$

Here the maximum temperatures for stationary ($\theta^{(s)}_{0,\max}$) and fast ($\theta^{(f)}_{0,\max}$) heating of the half-space are defined by formulas (3.7) and (4.7), respectively.

Later, Greenwood [10] modified formula (5.1) by summing the quantities that are the reverse of the squared maximum temperatures:

$$1/\theta^2_{0,\max} = 1/(\theta^{(s)}_{0,\max})^2 + 1/(\theta^{(f)}_{0,\max})^2. \tag{5.2}$$

Numerical analysis showed [10] that the results obtained using empirical formula (5.2) almost coincide with the values of the maximum temperature found from the exact solution [4].

In the case of a piecewise-homogeneous half-space, by analogy with formulas (5.1) and (5.2), we write

$$1/\theta_{1,\max} = 1/\theta^{(s)}_{1,\max} + 1/\theta^{(f)}_{1,\max}; \tag{5.3}$$

$$1/\theta^2_{1,\max} = 1/(\theta^{(s)}_{1,\max})^2 + 1/(\theta^{(f)}_{1,\max})^2. \tag{5.4}$$

There is another method of interpolating between the stationary and fast heating regimes. For $0 \leqslant \mathrm{Pe}_1 \leqslant 0.4$, according to (3.8), we have $\theta^{(s)}_{0,\max} = 1/\sqrt{\pi}$, and for $\mathrm{Pe}_1 > 20$, from relation (4.8) we obtain $\theta^{(f)}_{0,\max} = 1/\sqrt{\pi\,\mathrm{Pe}_1}$. Then,

TABLE 1

| $\mathrm{Pe}_1 = \mathrm{Pe}_2$ | $\theta^{(s)}_{1,\max}$ in (3.8) | $\theta^{(f)}_{1,\max}$ in (4.7) | $\theta_{1,\max}$ in (5.3) | $\theta_{1,\max}$ in (5.4) | $\theta_{1,\max}$ in (5.5) |
|---|---|---|---|---|---|
| 0.1 | 0.91 | — | 0.80 | 0.90 | 0.89 |
| 0.4 | 0.91 | — | 0.71 | 0.88 | 0.84 |
| 1 | — | — | 0.62 | 0.83 | 0.77 |
| 5 | — | — | 0.42 | 0.59 | 0.51 |
| 10 | — | — | 0.32 | 0.44 | 0.39 |
| 20 | — | 0.32 | 0.24 | 0.30 | 0.28 |
| 30 | — | 0.25 | 0.19 | 0.24 | 0.23 |
| 50 | — | 0.18 | 0.15 | 0.17 | 0.17 |

for the intermediate values of the Peclet number ($0.4 \leqslant \mathrm{Pe}_1 \leqslant 20$), the maximum temperature of a homogeneous half-space can be determined from the approximate formula

$$\theta_{0,\max} = 1/\sqrt{\pi(1 + \mathrm{Pe}_1)}\,,$$

which for $\mathrm{Pe}_1 \to 0$ and $\mathrm{Pe}_1 \gg 1$ agrees with the results given above.

Similarly, in the case of a piecewise-homogeneous half-space, we assume that

$$\theta_{1,\max} = \theta_{0,\max}\Big[1 - \frac{2C}{1+C} F^*(C, \delta_p, 0)\Big]. \tag{5.5}$$

The value of the function $F^*(C, \delta_p, 0)$ is calculated by formula (3.8) for

$$C = \frac{K^* - \lambda}{K^* + \lambda}, \qquad \lambda = \sqrt{\frac{1 + \mathrm{Pe}_1}{1 + \mathrm{Pe}_2}} \tag{5.6}$$

and with replacement of the dimensionless layer thickness $\delta$ by the complex $\delta_p = \delta^2\sqrt{1 + \mathrm{Pe}_1}$.

Table 1 gives calculated maximum dimensionless temperatures of a piecewise-homogeneous half-space for various values of the Peclet number $\mathrm{Pe}_1 = \mathrm{Pe}_2$ for $\delta = 0.1$ and $K^* = 0.5$. The data show that for the intermediate values of the Peclet number, the best agreement is obtained for the maximum temperatures found from Greenwood's formula (5.4) and using the procedure proposed in the present paper [formulas (5.5) and (5.6)]. This is also supported by the calculation results presented in Figs. 6 and 7.

**6. Results and Conclusions.** It was established that for the intermediate (moderate) values of the Peclet number, the effect of the layer on the maximum temperature of the compound body is determined by two dimensionless parameters — $C$ and $\delta_p$ in (5.6). The parameter $-1 < C < 1$, in turn, depends on the thermophysical properties of the materials of the layer and the foundation, the geometrical dimensions of the heating region, and the velocity of its motion. The case $C < 0$ corresponds to an increase and the case $C > 0$ to a decrease in the maximum temperature of the piecewise-homogeneous half-space compared to the homogeneous half-space; in the case where the materials of the layer and half-space are identical, $C = 0$.

The parameter $\delta_p$ depends on the relative layer thickness $\delta$ and the Peclet numbers $\mathrm{Pe}_1$. For small values of $\delta_p$, the function $F(C, \delta_p, 0) \approx 1$, and from relation (5.5) we obtain the following estimate for the maximum temperature of the compound body:

$$\theta_{1,\max} = \theta_{0,\max}\,\frac{1 - C}{1 + C} = \theta_{0,\max}\,\frac{\lambda}{K^*}. \tag{6.1}$$

[The parameter $\lambda$ is calculated by formula (4.1), and the relative thermal conductivity of the half-space and the layer $K^*$ by formula (1.6).] Converting to the dimensional quantities in formula (6.1) and taking into account that $\lambda \to 1$ as $\mathrm{Pe}_j \to 0$ ($j = 1, 2$), we obtain the maximum temperature of the homogeneous (made of the foundation material) half-space:

$$T_{1,\max} = (q_0 a/K_2)\theta_{0,\max}.$$

Hence, for small values of the layer thickness and $\mathrm{Pe}_1$, the effect of the layer on the maximum temperature can be ignored.

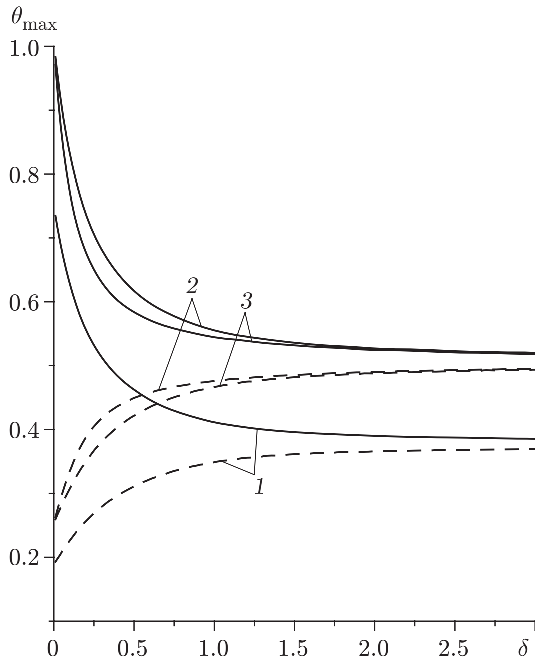


Fig. 6

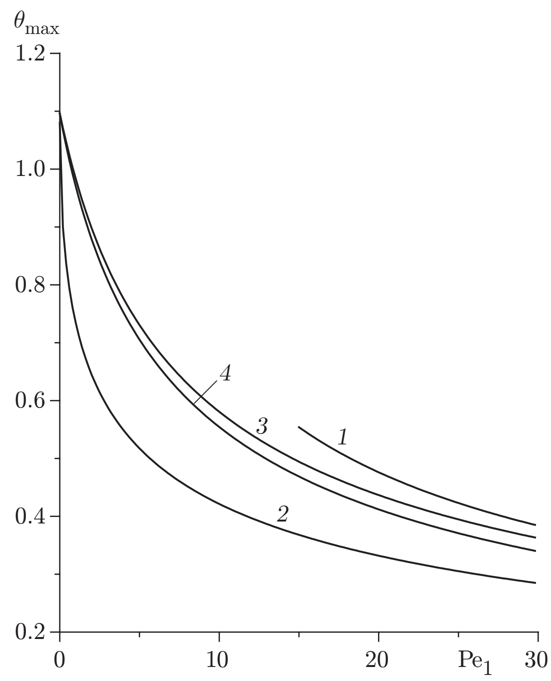


Fig. 7

Fig. 6. Dimensionless maximum temperature $\theta_{max}$ versus relative layer thickness $\delta$ at $Pe_1 = Pe_2 = 1$: the solid and dashed curves refer to $K^* = 0.5$ and 2, respectively; curves 1, 2, and 3 refer to calculations using formulas (5.3), (5.4), and (5.5), respectively.

Fig. 7. Dimensionless maximum temperature $\theta_{max}$ versus Peclet number ($Pe_1 = Pe_2 = 1$) for $K^* = 0.5$: curves 1–4 refers to calculations using formulas (4.7), (5.3), (5.4), and (5.5), respectively.

For large values of $\delta_p$ and $Pe_1$, we obtain function $F(C, \delta_p, 0) \approx 0$. Then, the solutions for the piecewise-homogeneous semi-infinite body and the homogeneous (with thermophysical properties of the layer) half-space coincide:

$$T_{1,\max} = (q_0 a/K_1)\theta_{0,\max}.$$

The proposed technique for determining the parameter $\delta_p$ and the function $F(C, \delta_p, 0)$ by formulas (3.8) and (5.6) can be used to estimate the effect of the layer material on the maximum temperature of piecewise-homogeneous bodies. It is found that for larger values of the Peclet number ($Pe > 20$) and fixed layer thickness, the external heat flow is completely absorbed by the layer, i.e., a skin effect takes place, in which the thermophysical properties of the foundation material do not influence the maximum temperature of the compound body. The limiting value of the parameter $\delta_p$ for which the heat is entirely absorbed by the layer is equal to unity. Hence, the effective heating depth can be calculated by the formula

$$d = a/\sqrt{1 + Pe_1}, \qquad Pe_1 \geqslant 0.4.$$

From (1.6) and (5.5) it follows that the ratio of the maximum surface temperature of the piecewise-homogeneous body $T_{1,\max}$ to that of the homogeneous body $T_{2,\max}$ is

$$\alpha = \frac{T_{1,\max}}{T_{2,\max}} = \frac{K^*}{\lambda}\Big[1 - \frac{2C}{1 + C}\, F^*(C, \delta_p, 0)\Big]. \tag{6.2}$$

If the layer thickness and the Peclet numbers are small enough, then, using relation (6.1) from formula (6.2), we obtain $\alpha \to 1$. At the same time, for large and intermediate values of the layer thickness and the Peclet number, the quantity $\alpha$ can be estimated by the approximate formula $\alpha \approx K^*/\lambda$ [the parameter $\lambda$ is calculated by formulas (4.1) for $Pe_1 > 20$ and formulas (5.6) for $0.4 \leqslant Pe_1 \leqslant 20$].

The function $F^*(C, \delta_p, 0)$ plays an important role in determining the maximum temperature not only on the layer surface but also at the interface between the materials of the conjugate bodies. We denote by $\beta = T_{2,\max}|_{z=d}/T_{1,\max}|_{z=0}$ the ratio of the maximum temperatures of the foundation and the layer. If $F^*(C, \delta_p, 0) \approx 1$, then, as noted above, the maximum temperature of the compound body is determined only by the thermophysical properties of the foundation and, hence, $\beta \to 1$. If $F^*(C, \delta_p, 0) \approx 0$, then the maximum temperature is affected primarily by the properties of the layer material and $\beta \to 0$.

Numerical analysis of the problem showed that when the shape of the heating region changes from square to circular with the same surface area, the maximum temperature of the compound body changes only slightly (by not more than 5%).

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